MOTION OF A FLUID TOWARD A BOREHOLE FILTER OF FINITE LENGTH IN A VERY THICK LAYER

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1. The differential equation of filtration in curvilinear orthogonal coordinates corresponding to the flow under consideration. Let us assume that potential motion of a liquid or gas occurs in some region of space occupied by a porous medium. We choose a curvilinear orthogonal u-, v-, and w-coordinate system in which each coordinate surface of a single family, e.g., u = const, coincides at a given instant with one of the quipotential surfaces φ = const, where $\varphi(u, t)$ is the potential of the mass filtration rate averaged over the area of the equipotential surface F(u), i.e., of the mass filtration rate corresponding to some average values of the coordinates v_* and w_*.

Let us consider a layer element of the porous medium between the areas F(u) and F(u) + (dF/du)du. We use a procedure common in derivation of the continuity equation to find the mass which accumulates in the layer in the time interval dt,

$$F(u)\frac{\partial\varphi}{\partial u}dt-\left[F(u)+\frac{dF}{du}du\right]\left(\frac{\partial\varphi}{\partial u}+\frac{\partial^2\varphi}{\partial u^2}du\right)dt=$$
$$=-\left[\frac{dF}{du}\frac{\partial\varphi}{\partial u}du+\frac{dF}{du}\frac{\partial^2\varphi}{\partial u^2}(du)^2+F(u)\frac{\partial^2\varphi}{\partial u^2}du\right]dt.$$

Omitting the term of the highest order of smallness, we obtain the following expression for the accumulated mass:

$$-\left[\frac{dF}{du}\frac{\partial\varphi}{\partial u}+F(u)\frac{\partial^2\varphi}{\partial u^2}\right]du\,dt. \tag{1.1}$$

This mass can also be expressed as

$$F(u)\frac{\partial(m\rho)}{\partial t}du\,dt, \tag{1.2}$$

where ρ is the density of the liquid (or gas).

Equating (1.1) to (1.2), we obtain the required equation

$$\frac{dF}{du}\frac{\partial\varphi}{\partial u}+F(u)\frac{\partial^2\varphi}{\partial u^2}+F(u)\frac{\partial(m\rho)}{\partial t}=0. \tag{1.3}$$

For steady flow we have

$$\frac{dF}{du}\frac{d\varphi}{du}+F(u)\frac{d^2\varphi}{du^2}=0. \tag{1.4}$$

We express the mass filtration rate as

$$\partial\varphi/\partial u=\rho\partial\Phi/\partial u, \tag{1.5}$$

where Ψ is the potential of the filtration rate averaged over the area F(u). Substituting the value of $\partial\varphi/\partial u$ into Eq. (1.3) we obtain

$$\left[\rho\frac{dF}{du}+F(u)\frac{\partial\rho}{\partial u}\right]\frac{\partial\Phi}{\partial u}+\rho F(u)\frac{\partial^2\Phi}{\partial u^2}+F(u)\frac{\partial(m\rho)}{\partial t}=0. \tag{1.6}$$

We examine the case of an elastic fluid whose density ρ as a function of pressure p can be expressed approximately as

$$\rho/\rho_0\approx 1+\beta_1(p-p_0),\quad \beta_1(p-p_0)\ll 1, \tag{1.7}$$

where ρ_0 is the density at atmospheric pressure p_0; β_1 is the volume-elasticity coefficient of the fluid.

We assume that the second term of the expression in brackets in (1.6) is negligibly small as compared with its first term. Equation (1.6) becomes

$$\rho\left[\frac{dF}{du}\frac{\partial\Phi}{\partial u}+F(u)\frac{\partial^2\Phi}{\partial u^2}\right]+F(u)\frac{\partial(m\rho)}{\partial t}=0. \tag{1.8}$$

This equation can be used to solve the problem of nonplanar motion of an elastic fluid in an elastic layer.

2. Steady filtration in an ellipsoidal axisymmetric field. Let the borehole expose an infinite layer. The filter in the cylindrical borehole is of length 2h and extends over a straight segment of the borehole axis. We assume that the borehole filter is a prolate ellipsoid of revolution of focal length 2h and that the filter wall is an equipotential surface. The mass discharge of the borehole is M = const.

Under these conditions the equipotential surfaces are confocal ellipsoids of revolution with semiaxes α, β, and γ,

$$\alpha=h\operatorname{ch}u,\ \beta=\gamma=h\operatorname{sh}u. \tag{2.1}$$

Where u is the value of the degenerate ellipsoidal coordinate which determines the equipotential surface u = const.

The coordinate u and the potential φ are related by

$$dM/dF=\partial\varphi/\partial u, \tag{2.2}$$

where dM is the mass flow rate through the area dF.

Separating variables in (2.2) and integrating, we obtain the mass flow rate M through the entire area F(u) of the equipotential surface

$$M=\iint\limits_{(F)}\frac{\partial\varphi}{\partial u}dF=\frac{d\varphi}{du}F(u)=q(u,v_*,w_*)F(u) \tag{2.3}$$

Where $d\varphi/du = q(u, v^*, w=)$ is the mass filtration rate averaged over the area F and corresponding to some average values of the coordinates v_* and w_*.

From (2.3) we obtain

$$d\varphi/du=M/F(u), \tag{2.4}$$

where

$$F(u)=2\pi\alpha\beta\left(\sqrt{1-\varepsilon^2}+\frac{\arcsin\varepsilon}{\varepsilon}\right)\left(\varepsilon=\frac{\sqrt{\alpha^2-\beta^2}}{\alpha}\right), \tag{2.5}$$

$$F(u)=4\pi h^2\xi(u),\ \xi(u)=$$
$$={}^1/_2\operatorname{sh}u\operatorname{ch}u(\operatorname{th}u+\operatorname{ch}u\arcsin(1:\operatorname{ch}u). \tag{2.6}$$

Equation (2.4) can be regarded as the first integral of Eq. (1.4). Its solution is

$$\varphi=\frac{M}{4\pi h^2}\int\frac{du}{\xi(u)}+C\qquad(C=\text{const}). \tag{2.7}$$

Let us replace the coordinate u by the new dimensionless variable r/h according to the condition

$$(r/h)^j=\xi(u)\qquad(1\leqslant j=\text{const}\leqslant 2). \tag{2.8}$$

Here r is expressed in units of length. We find from (2.5) that

$$F(u)=4\pi h^{2-j}r^j=F_1(r). \tag{2.9}$$

With Eqs. (2.4), (2.6), and (2.8), we write

$$d\varphi/du=M/4\pi h^{2-j}r^j=d\varphi^*/dr, \tag{2.10}$$

where φ^* is some function corresponding to the potential φ.

With Eqs. (2.9) and (2.10) we alter Eq. (1.4). We have

$$\frac{dF(u)}{du}\frac{d\varphi}{du}=4\pi h^2\frac{j}{h}\left(\frac{r}{h}\right)^{j-1}\frac{d\varphi^*}{dr}\frac{dr}{du},$$

$$F(u)\frac{d^2\varphi}{du^2}=F_1(r)\frac{d^2\varphi^*}{dr^2}\frac{dr}{du}=4\pi h^2\left(\frac{r}{h}\right)^j\frac{d^2\varphi^*}{dr^2}\frac{dr}{du}. \tag{2.11}$$

Substituting Eq. (2.11) into Eq. (1.4), we obtain

$$\frac{d^2\varphi^*}{dr^2} + \frac{j}{r}\frac{d\varphi^*}{dr} = 0 \quad (\text{for } r \neq 0). \tag{2.12}$$

The solution of Eq. (2.12) is

$$\varphi^* = \frac{M}{4\pi h(1-j)}\left(\frac{r}{h}\right)^{1-j} + C^* \qquad (C^* = \text{const}) \tag{2.13}$$

(see (2.10)).

We see from Eqs. (2.5) and (2.8) that in the case j = 1 the area of any equipotential surface r = const is equal to the area of the side surface of a cylinder of height h and radius r; Eq. (2.12) is simply the equation of plane radial flow in cylindrical coordinates. The other extreme case j = 2 corresponds to spherically radial flow and a spherical coordinate system; the area of the equipotential surface r = const in this case is $4\pi r^2$ (see (2.5) and (2.8)).

$\frac{r_b}{h}$	j	$\eta\left(\frac{r_b}{h}, \frac{r_c}{r_b}\right)$	$\frac{r_c}{r_b}$	$\eta\left(\frac{r_b}{h}, \infty\right)$
0.1	1.103	1.24	12.95	5.300
0.01	1.053	1.12	93	5.225
0.001	1.035	1.08	750	5.225

When the length of the borehole filter is rather large as compared with the radius r_b, e.g., $h \geq 10r_b$ we can quite readily determine the value of the constant j under the condition $1 < j < 2$. We limit ourselves to these cases. Equations (2.5) and (2.8) yield the values of $\xi(u_b)$ at the borehole wall,

$$\xi(u_b) = r_b/h \text{ for } j = 1,\ \xi(u_b) = (r_b/h)^2 \text{ for } j = 2.$$

We therefore have

$$\xi(u_c) = \left(\frac{r_c}{h}\right)^j, \quad \text{or} \quad j = \frac{\lg \xi(u_c)}{\lg(r_c/h)} \tag{2.14}$$

at the borehole wall for $1 < j < 2$.

We determine $\xi(u_b)$ from Eq. (2.6), setting $\operatorname{sh} u_b \approx r_b/h \leq 0.1$.

Stipulating the conditions at the boundaries of some region of space between the feed surface and the borehole, we obtain a formula for the mass discharge of the borehole.

Let $\varphi^* = \varphi_b^*$ for $r = r_b$ and $\varphi^* = \varphi_c^*$ for $r = r_c$, where r_c is the coordinate of the feed surface. From (2.13) we obtain

$$M = \frac{4\pi h^{2-j}(1-j)(\varphi_c^* - \varphi_b^*)}{r_c^{1-j} - r_b^{1-j}} = \frac{4\pi h^{2-j} r_b^j (1-j)(\varphi_c^* - \varphi_b^*)}{r_b[(r_c/r_b)^{1-j} - 1]}. \tag{2.15}$$

Noting that (by (2.5) and (2.9)) the quantity

$$4\pi h^{2-j} r_b^j = 4\pi h^2 \xi(u_b) = F(u_b)$$

gives the area, we use (2.14) to transform (2.15) into

$$M = \frac{4\pi h(\varphi_c^* - \varphi_b^*)}{\eta(r_b/h, r_c/r_b)\ln(h/r_b)}, \tag{2.16}$$

$$\eta\left(\frac{r_b}{h}, \frac{r_c}{r_b}\right) = \frac{r_b}{h\xi(u_b)}\left[1 - \left(\frac{r_c}{r_b}\right)^{1-j}\right] : \ln\frac{r_b}{h\xi(u_b)}. \tag{2.17}$$

If r_c is so large that we can set $(r_c/r_b)^{1-j} = 0$, we have

$$\eta\left(\frac{r_b}{h}, \infty\right) = \frac{r_b}{h\xi(u_b)} : \ln\frac{r_b}{h\xi(u_b)}.$$

Let us imagine a layer bounded above by an impermeable horizontal stratum and which occupies the lower half-space. Let the stratum be exposed by a vertical borehole sunk to a depth h into the layer.

The known formulas for the mass discharge M_1 of a vertical borehole in a semiinfinite layer allow us to consider such a borehole equivalent to a hydrodynamically perfect borehole in a layer of thickness h with a feed surface of radius νh, where $1 < \nu < 2$. Polubarinova-Kochina [1] proposed the following formula for the discharge M_1 of a vertical borehole in a semiinfinite layer:

$$M_1 = \frac{2\pi h(\varphi_c - \varphi_b)}{\ln(\sqrt{3}h/r_b)} \qquad (\nu = \sqrt{3}). \tag{2.18}$$

In this case the discharge can also be computed with the aid of formula (2.16), in which the coefficient 4 must be replaced by the coefficient 2 (the half-space belongs to the layer). The discharges calculated by means of Eqs. (2.16) and (2.18) coincide if

$$\eta(r_b/h, r_c/r_b)\ln(h/r_b) = \ln(\sqrt{3}h/r_b). \tag{2.19}$$

Equation (2.19) defines in this case the coordinate r_c of the feed surface for that region of the layer exposed by the borehole.

The coordinate r_c can be found from (2.19) with the aid of (2.17). The table below enables us to compare the quantities j, $\eta(r_b/h, r_c/r_b)$, and r_c/r_b for

$$r_b/h = 0.1,\ 0.01,\ \text{and}\ 0.001$$

in the case of a borehole exposing an infinitely deep layer to a depth h; it is equivalent to a hydrodynamically perfect borehole in a layer of thickness h and with a radius of the feed surface equal to $(3)^{1/2}h$ (see condition (2.19)). The table also contains values of $\eta(r_b/h, \infty)$ corresponding to the case of an infinitely distant feed surface. The computations were completed with Eqs. (2.14), (2.17), and (2.19).

Only for $r_b/h = 0.1$ does the coordinate r_k exceed the length h of the filter; in the remaining cases $r_c < h$. As $r_c \to \infty$ the discharge M can diminish by more than 4.8 times. (This follows from Eq. (2.16) and the table.)

3. A borehole of finite length in an unbounded layer under elastic conditions. Let us consider the motion of a fluid elastic mass in an unbounded elastic layer tapped by a borehole containing a filter of length 2h. We assume that the permeability k of the layer and the dynamic viscosity μ of the fluid are constant. This yields the following expression for the potential of the average filtration rate $d\Phi^*/dr$,

$$\Phi^* = kp/\mu + C. \tag{3.1}$$

Where p is the reduced pressure and C is some constant. Let the elastic properties be related by the expression

$$\partial m/\partial t = \beta_b \partial p/\partial t. \tag{3.2}$$

Where m is the porosity of the layer and β_b is its volume elasticity coefficient.

We represent the elastic properties of the liquid as

$$\partial\rho/\partial t = \rho\beta_1\,\partial p/dt, \tag{3.3}$$

where β_1 is the volume-elasticity coefficient of the liquid. With (3.2) and (3.3) we can write

$$\partial(m\rho)/dt = \rho(m\beta_1 + \beta_b)\,\partial p/\partial t. \tag{3.4}$$

In accordance with (2.9) and (3.1), Eq. (1.8), in which r is the principal coordinate, takes the form

$$\varkappa\left(\frac{\partial^2 p}{\partial r^2} + \frac{j}{r}\frac{\partial p}{\partial r}\right) = \frac{\partial p}{\partial t}$$

$$\left(\varkappa = \frac{k}{\mu\beta^*}, \quad \beta^* = m\beta_1 + \beta_b\right). \tag{3.5}$$

Here $\varkappa$ is the piezoconductivity coefficient of the layer, and β^* is the volume-elasticity coefficient.

Equation (3.5) can be interpreted as the heat-conduction equation in a (j + 1)-dimensional space. The solution of this equation can be constructed by analogy with an instantaneous sink in a (j + 1)-dimensional space [2]. A similar analogy was used to solve the equation corresponding to degenerate isotropic turbulence for very small pulsation rates [3–5]. In the case of an instantaneous sink the solution of Eq. (3.5) can be written as

$$p(r, t) = C_1 - C_2 t^{-1/2(j+1)} \exp(-r^2/4\varkappa t). \qquad (3.6)$$

Let us determine the constants C_1 and C_2. Let $p = p_0$ for $t = 0$. Yet for $r > 0$ we have

$$\lim_{t\to 0}\left[t^{-1/2(j+1)} \exp\left(-\frac{r^2}{4\varkappa t}\right)\right] = 0. \qquad (3.7)$$

Hence, $C_1 = p_0$.
We can therefore write

$$\Delta p = p_0 - p(r, t) = C_2 t^{-1/2(j+1)} \exp(-r^2/4\varkappa t). \qquad (3.8)$$

The volume $d\tau_1$ of fluid emerging from an ellipsoidal layer element $d\tau = 4\pi h^{2-j} r^j dr$ is

$$d\tau_1 = \beta^* \Delta p \, d\tau = 4\pi h^{2-j} \beta^* C_2 t^{-1/2(j+1)} \exp\left(-\frac{r^2}{4\varkappa t}\right) r^j dr. \qquad (3.9)$$

The fluid volume τ_1 discharged from the entire layer can be determined through integration of (3.9) [6]

$$\tau_1 = 4\pi h^{2-j} \beta^* C_2 t^{-1/2(j+1)} \int_0^\infty r^j \exp\left(-\frac{r^2}{4\varkappa t}\right) dr =$$

$$= 2\pi h^{2-j} \beta^* C_2 (4\varkappa)^{1/2(j+1)} \Gamma(1/2(j+1)). \qquad (3.10)$$

Here Γ is the gamma function.
Taking account of the relationship between $\varkappa$ and β^*, we determine from (3.10) that

$$C_2 = \frac{\tau_1 \mu}{2^{2+j} \pi k \varkappa^{1/2(j-1)} h^{2-j} \Gamma(1/2(j+1))}. \qquad (3.11)$$

Now let us turn to the case of a borehole, with a filter of length 2h, which taps an infinite layer at the initial instant and acts continuously with a constant volume discharge Q. If an instantaneous sink exists at some instant t, we can rewrite formula (3.8) as

$$\Delta p = \frac{\tau_1 \mu}{2^{2+j} \pi \varkappa^{1/2(j-1)} k h^{2-j} \Gamma(1/2(j+1))} (t-t')^{-1/2(j+1)} \times$$

$$\times \exp\left(-\frac{r^2}{4\varkappa(t-t')}\right). \qquad (3.12)$$

Let the sink exist over the time interval dt'.
The volume of fluid discharged from the layer during the existence of the sink is given by

$$d\tau_1 = Q\, dt'. \qquad (3.13)$$

Recalling (3.12) and (3.13), we obtain a formula for the pressure drop Δp in the layer during continuous operation of the borehole from $t' = 0$ to $t' = t$

$$\Delta p = \frac{Q\mu}{2^{2+j} \pi h^{2-j} k \Gamma(1/2(j+1)) \varkappa^{1/2(j-1)}} \int_0^t (t-t')^{-1/2(j+1)} \times$$

$$\times \exp\left[-\frac{r^2}{4\varkappa(t-t')}\right] dt'. \qquad (3.14)$$

Substituting $1/(t - t') = \theta$, we can rewrite Eq. (3.14) as

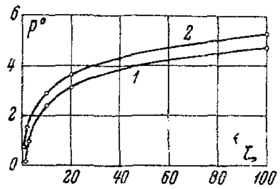
$$\Delta p = \frac{Q\mu}{2^{2+j} \pi h^{2-j} k \varkappa^{1/2(j-1)} \Gamma(1/2(j+1))} \times$$

$$\times \int_{\theta_0}^\infty \theta^{1/2(j-3)} \exp\left(-\frac{r^2}{4\varkappa}\theta\right) d\theta =$$

$$= \frac{Q\mu}{8\pi k h \Gamma(1/2(j+1))} \left(\frac{r}{h}\right)^{1-j} \Gamma\left(\frac{j-1}{2}, \frac{r^2}{4\varkappa t}\right),$$

$$\theta_0 = \frac{1}{t}, \quad \Gamma\left(\frac{j-1}{2}, \frac{r^2}{4\varkappa t}\right) =$$

$$= \int_R^\infty x^{1/2(j-3)} e^{-x} dx, \quad \left(R = \frac{r^2}{4\varkappa t}\right). \qquad (3.15)$$



Here $(j - 1)/2$, $r^2/4\varkappa t)$ is an incomplete gamma function [6],

$$\frac{\Gamma(1/2(j-1), r^2/4\varkappa t)}{\Gamma(1/2(j+2))} =$$

$$= \frac{2}{j-1} - \frac{\gamma(1/2(j-1), r^2/4\varkappa t)}{\Gamma(1/2(j+1))}, \qquad (3.16)$$

where

$$\gamma = \left(\frac{j-1}{2}, \frac{r^2}{4\varkappa t}\right) = \int_0^R x^{\frac{1}{2}(j-3)} e^{-x} dx \quad \left(R = \frac{r^2}{4\varkappa t}\right).$$

With the familiar recursion formula for the gamma function, we obtain [7]

$$\frac{\gamma(1/2(j-1), r^2/4\varkappa t)}{\Gamma(1/2(j+1))} =$$

$$= \frac{2}{j-1}\left[1 - P\left(\frac{r^2}{2\varkappa t}, j-1\right)\right],$$

$$P\left(\frac{r^2}{2\varkappa t}, j-1\right) = \frac{1}{2(1/2(j-3)) \Gamma(1/2(j-1))} \times$$

$$\times \int_{R^*}^\infty x^{j-2} \exp\left(-\frac{x^2}{2}\right) dx \quad \left(R^* = \frac{r^2}{\sqrt{2\varkappa t}}\right). \qquad (3.17)$$

On the basis of (3.17) from (3.15) and (3.16), we find that

$$\Delta p = \frac{Q\mu}{4\pi k h}\left(\frac{r}{h}\right)^{1-j} \frac{P(r^2/2\varkappa t, j-1)}{j-1} =$$

$$= \frac{Q\mu}{8\pi k h} f\left(\frac{r}{h}, \frac{r^2}{2\varkappa t}\right),$$

$$f\left(\frac{r}{h}, \frac{r^2}{2\varkappa t}\right) = \frac{2}{j-1}\left(\frac{r}{h}\right)^{1-j} P\left(\frac{r^2}{2\varkappa t}, j-1\right). \qquad (3.18)$$

Let us compare the pressure drops at the walls of two boreholes, one of which operates with a constant discharge in a layer of thickness 2h, while the other is equipped with a filter of finite length 2h and taps an infinitely thick layer with the same discharge. The motion in the former case is plane-radial, and in the latter case it is three-dimensional and axisymmetric.

The pressure drop Δp_b at the borehole walls in the first case is, as we know, proportional to the integral exponential function $-E_i(-r_b^2/4\varkappa t)$; in the second case it is proportional to the function $f(r_b/h, r_b^2/4\varkappa t)$ (see (3.18)). To find the corresponding values of these functions it is sufficient to complete the necessary comparison.

The curves of the functions $-\mathrm{Ei}(r_b^2/4\varkappa t)$ and $f(r_b^2/h, r_b^2/2\varkappa t)$ appear in the figure. The axis of abscissas represents the dimensionless parameter $\zeta = 2\varkappa t/r_b^2$ and the axis of ordinates represents the dimensionless quantity $p^0 = (8\pi kh/Q\mu)\,\Delta p_b$ equal to the function $-\mathrm{Ei}(-r_b^2/4\varkappa t)$ in the case of curve 1 and to the function $f(r_b/h, r_b^2/2\varkappa t)$ in the case of curve 2. We assume that the quantity r_b/h is 0.1, which (by (2.14)) means that $j \approx 1.1$.

The curves reflect the greater intensity of the pressure-drop process for a borehole in an infinitely thick layer as compared with the same process for a perfect borehole in a layer of finite thickness. We see that the difference between the intensities is not large.

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